L-functions: structure and tools

David Farmer
AIM

joint work with
Sally Koutsoliotas and Stefan Lemurell
and
Ameya Pitale, Nathan Ryan, and Ralf Schmidt

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What is an L-function?

"A Dirichlet series which has a functional equation and an Euler product."

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Can you be more specific?

The Selberg class of L-functions

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with $a_n = O_\epsilon(n^\epsilon)$.

$$\begin{aligned}\Lambda(s) &:= Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i) \cdot L(s) \\ &= \varepsilon \overline{\Lambda}(1-s)\end{aligned}$$

where $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \geq 0$, and $|\varepsilon| = 1$.

$$\log L(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with $b_n = 0$ unless n is a positive power of a prime, and $b_n \ll n^\theta$ for some $\theta < 1/2$.

Issues with Selberg's formulation

- The Ramanujan bound $a_n = O_\varepsilon(n^\varepsilon)$ has not been shown to hold for most L-functions, so most L-functions are not known to be in the Selberg class.
- The parameters in the functional equation are not well defined (because of the duplication formula of the Γ-function).
- Euler product? Where is the product?

While it is an interesting challenge to show that such L-functions only arise from known sources, it can be helpful to formulate more restrictive axioms.

We use the term *analytic L-function* for functions which satisfy the following axioms.

The Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\sigma > 1$.

The functional equation

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

$$\Lambda(s) = N^{s/2} \prod_{j=1}^{d_1} \Gamma_{\mathbb{R}}(s+\mu_j) \prod_{j=1}^{d_2} \Gamma_{\mathbb{C}}(s+\nu_j) \cdot L(s)$$

satisfies

$$\Lambda(s) = \varepsilon \overline{\Lambda}(1-s).$$

N is the *conductor*
(d_1, d_2) is the *signature*
$d = d_1 + 2d_2$ is the *degree*

The analogue of the Selberg eigenvalue conjecture is:
$\Re(\mu_j) \in \{0, 1\}$
$\Re(\nu_k) \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$

The Euler product

There exists a Dirichlet character χ mod N, called the *central character* of the L-function, such that

$$L(s) = \prod_{p \text{ prime}} F_p(p^{-s})^{-1}, \tag{1}$$

is absolutely convergent for $\sigma > 1$, where F_p is a polynomial of the form

$$F_p(z) = 1 - a_p z + \cdots + (-1)^d \chi(p) z^d. \tag{2}$$

Ramanujan bound: Write F_p in factored form as

$$F_p(z) = (1 - \alpha_{1,p} z) \cdots (1 - \alpha_{d_p,p} z). \tag{3}$$

If p is good (i.e., $p \nmid N$, so $d_p = d$) then $|\alpha_j| = 1$.

Further conditions

Ramanujan bound at a bad prime: $|\alpha_{j,p}| = p^{-m_j/2}$ for some $m_j \in \{0, 1, 2, ...\}$, and $\sum m_j \le d - d_p$.

The degree of a bad factor can't be too small:

$$d_p + \mathrm{ord}_p(N) \ge d$$

Parity: The spectral parameters determine the parity of the central character:

$$\chi(-1) = (-1)^{\sum \mu_j + \sum (2\nu_j + 1)}.$$

Some consequences

- If π is a tempered cuspidal automorphic representation of $GL(n)$ then $L(s, \pi)$ is an analytic L-function. (There is also a version of the axioms which does not require π to be tempered.)
- The functional equation data are well defined.
- *Strong multiplicity one:* If two analytic L-functions have the same local factors for all but finitely many p, then they are identical. (In particular, an analytic L-function is determined by its good local factors.)
- *Exercise:* If an analytic L-function satisfies

$$\Lambda(s) = 60^{s/2}\Gamma_{\mathbb{C}}(s + \tfrac{1}{2})\Gamma_{\mathbb{C}}(s + \tfrac{3}{2})L(s) = \Lambda(1 - s)$$

then it must be primitive (i.e., not a product of lower degree analytic L-functions).

Tools: the explicit formula

Theorem. [Mestre] *Assuming various reasonable assumptions, the conductor N of a genus g curve satisfies $N > 10.323^g$.*

Strategy of the proof
Step1: Use the explicit formula to show that an analytic L-function with functional equation

$$\Lambda(s) = N^{s/2}\Gamma_{\mathbb{C}}(s+\tfrac{1}{2})^g L(s) = \pm\Lambda(1-s)$$

cannot exist unless $N > 10.323^g$.
Step 2: Note that the Hasse-Weil L-function of a genus g curve (conjecturally) satisfies such a functional equation. QED

Limitation of the method
Since there *does* exist an analytic L-function with $N = 11^g$, namely

$$L(s, E_{11.a})^g,$$

there is not much room for improvement using that strategy.

Tools: the approximate functional equation

Let $g(s)$ be a test function and suppose

$$\Lambda(s) = Q^s \prod_{j=1}^{a} \Gamma(\kappa_j s + \lambda_j) \cdot L(s) = \varepsilon \overline{\Lambda}(1-s)$$

is entire. Then under appropriate conditions:

$$\Lambda(s)g(s) = Q^s \sum_{n=1}^{\infty} \frac{a_n}{n^s} f_1(s,n) + \varepsilon Q^{1-s} \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} f_2(1-s,n)$$

where

$$f_1(s,n) = \frac{1}{2\pi i} \int_{(\nu)} \prod_{j=1}^{a} \Gamma(\kappa_j(z+s) + \lambda_j) z^{-1} g(s+z) (Q/n)^z dz$$

$$f_2(1-s,n) = \frac{1}{2\pi i} \int_{(\nu)} \prod_{j=1}^{a} \Gamma(\kappa_j(z+1-s) + \overline{\lambda_j}) z^{-1} g(s-z) (Q/n)^z dz$$

The Approximate Functional Equation

$g(s)$: a nice test function

$$g(s)\ \Lambda(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} f_{compl.}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} f_{compl.}(g)$$

Solving for $L(s)$ gives

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} h_{compl.}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} h_{compl.}(g)$$

Choose s_0 and $g_1(s)$:

$$L(s_0) = \bigstar + \bigstar\ a_2 + \bigstar\ a_3 + \bigstar\ a_4 + \bigstar\ a_5 + \bigstar\ a_6 + \ldots$$

Example: $g(s) = e^{\frac{7}{8}s}$, if $f \in S_{24}(\Gamma_0(1))$ then

$$L(\tfrac{1}{2}, f) = 1.473a_1 + 1.186a_2 - 0.0959a_3 - 0.00772a_4 + 0.000237a_5 + \cdots.$$

To show an L-function does not exist:

Choose a point s_0 and two test functions g_1 and g_2.

$$s_0, g_1 : L(s_0) = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \ldots$$

$$s_0, g_2 : L(s_0) = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \ldots$$

Subtracting gives:

$$0 = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \ldots$$

Check whether the Ramanujan bound for a_j is consistent with LHS = RHS.

Example: no hyperelliptic curve with $N = 125$ and $\varepsilon = +1$

Choose $s_0 = \frac{1}{2}$. Using the test function $g(s) = 1$:

$$L(\tfrac{1}{2}) = 0.123 + 0.0241a_2 + 0.0058a_3 + 0.00195a_4 + 0.00075a_5 + \cdots$$

and with $g(s) = e^{\frac{1}{4}s}$:

$$L(\tfrac{1}{2}) = 0.116 + 0.01834a_2 + 0.00444a_3 + 0.0013a_4 + 0.00044a_5 + \cdots .$$

Subtracting and rescaling:

$$0 = 1 + 0.427a_2 + 0.187a_3 + 0.087a_4 + 0.043a_5 + 0.022a_6 + \cdots .$$

Try $F_2(T) = 1 - 2\sqrt{2}T + 4T^2 - 2\sqrt{2}T^3 + T^4$:

$$0 = 2.57 + 0.254a_3 + 0.049a_5 + 0.0130a_7 + 0.00399a_9 + \cdots .$$

The Ramanujan bound is $|a_p| \leq 4$.
$\Longrightarrow$ Contradiction. (34 more choices to try for F_2.)

Exercise, part 2

Show that there is no analytic L-function satisfying

$$\Lambda(s) = 60^{s/2}\Gamma_{\mathbb{C}}(s+\tfrac{1}{2})\Gamma_{\mathbb{C}}(s+\tfrac{3}{2})L(s) = \pm\Lambda(1-s).$$

If '60' is replaced by '61' then there is such an L-function, and it is primitive. There is a non-primitive example with $N = 55$, so the explicit formula is unlikely to be helpful.

Evaluating L-functions

Convenient test functions are: $g(s) = \exp(i\beta s + \alpha(s-\gamma)^2)$.

Valid choice if $\alpha = 0$ and $|\beta| < \pi d/4$,
or $\alpha > 0$ and any β.

The contribution of the n^{th} Dirichlet coefficient is approximately

$$\star_n \approx \exp(c(n/\sqrt{N})^{2/d})$$

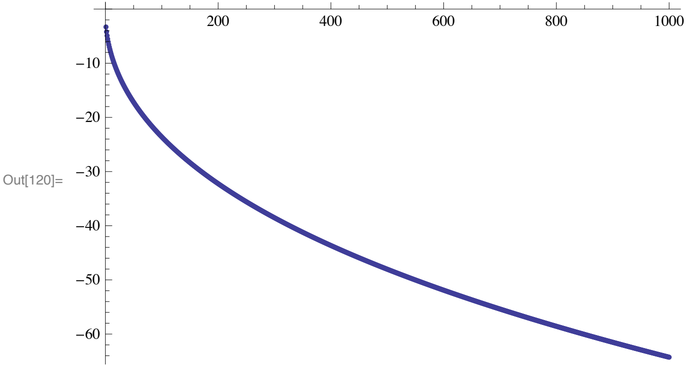
so

$$\log(\star_n) \approx -C\, n^{2/d}$$

for some $C > 0$.

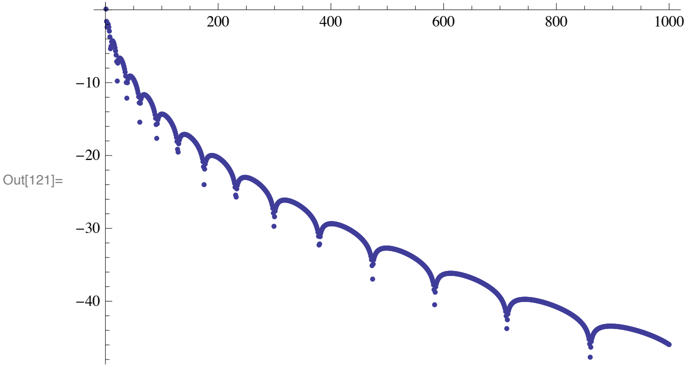
Evaluating L-functions

Degree 4, $\alpha = 0$, $\beta = 0$



Evaluating L-functions

Degree 4, $\alpha = 0$, $\beta = 2$



Evaluating L-functions

Observation [F and Nathan Ryan] You can "average" the separate evaluations to obtain a surprisingly small error: much smaller than square-root cancellation due to randomness.

Let g_β be suitable test functions and $L_\beta(s_0)$ the evaluation of $L(s)$ at s_0 using the test function g_β.

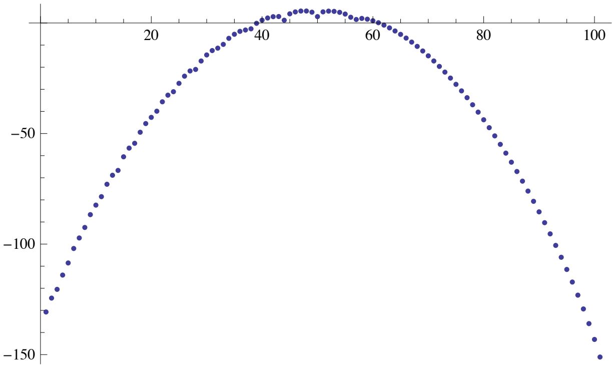
If $\sum c_\beta = 1$ then $L(s_0) = \sum_\beta c_\beta L_\beta(s_0)$.

With $g_\beta(s) = \exp(i\beta s + (s - 100i)^2/100)$, for
$\beta = -30/20, -59/20, \ldots, 70/20$,
and using *no coefficients at all*, we find

$$Z(\tfrac{1}{2} + 100i, \Delta) =$$
$$-\,0.23390\,65915\,56845\,20570\,65824\,17137\,27923\,81141\,00783$$
$$\pm\,3.28 \times 10^{-42}.$$

Evaluating L-functions

$\log(c_\beta)$, $\beta = -30/20, -59/20, \ldots, 70/20$.



Evaluating L-functions

What if there is more than one L-function with the given functional equation?

For $f \in S_{24}(\Gamma(1))$,

$$Z(\tfrac{1}{2} + 100i, f) = $$
$$1.87042\,65340\,29268\,89914\,33391\,93910\,89610\,35060\,87410\;a_1$$
$$+\,1.12500\,88863\,02338\,48447\,34844\,21487\,86375\,36206\,60254\;a_2$$
$$\pm\, C_f \times 2.86 \times 10^{-43}.$$

where C_f satisfies $|a_n| \le C_f\, d(n)$.

More tools

Sato-Tate group: Conjecturally, each L-function is associated to a subgroup of $L_{ST} \subset U(d)$, called the Sato-Tate group of the L-function, such that the local factors $F_p(T)$ have the same distribution as the characteristic polynomials of Haar-random matrices from L_{ST}.

Selberg orthonormality conjecture: If $L_1(s) = \sum a_n n^{-s}$ and $L_2(s) = \sum b_n n^{-s}$ are primitive L-functions, then

$$\langle a_p \overline{b_p} \rangle = \delta_{L_1, L_2}.$$

Rankin-Selberg and other operations: If $L_1(s)$ and $L_2(s)$ are analytic L-functions, then conjecturally so are $(L_1 \otimes L_2)(s)$ and $L_1(s, \mathrm{sym}^n)$. The Sato-Tate group of an L-function determines which operations give entire functions.

Some questions

1. Prove that there are only finitely many L-functions with a given functional equation.

2. Prove that if two L-functions have sufficiently many initial Dirichlet coefficients in common, then they must be equal. Find a useful effective bound.

3. Prove that a primitive analytic L-function must equal $L(s, \pi)$ for some appropriate π.

Note that the last problem has only been solved for degree $d < 2$, and for a small number of conductors when $d = 2$.